MMPLS

Tai-Hsuan Chung

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On the Existance of Flips

by Hacon & M^cKernan

Outline of this talk

§1. Main thm + Cor

§2. Some history

§3. f.g. of R

§4. Main technical thm

§1. Main thm & Cor

boundary Δ is $\mathbb{R}$-divisor

Main thm Assume the real MMP in dim $n-1$,
then flips exist in dim n.

Cor Assume termination of real flips in dim $n-1$,
then flips exist in dim n.

Pf follows from $\exists$ real flips in dim 3.

Cor Flips exist in dim 4

Pf clear as Shokurov proved termination of real flips in dim 3.

§ 2. Some history

- Kulikov is the first to use codim 2 surgery systematically w/ the aim of constructing minimal models.

- In mid '80s Kawamata, Mori, Shokurov, Tsunoda all independently showed existence of semistable 3-fold flips.

  $\exists$ 3-fold flips in general

- In early '90s, Shokurov discovered a new approach to existence of 3-fold flips.

  Shokurov's strategy consists of two stages:

  ① reduction of flips to pl flips

  ② 3-fold pl flips were constructed by a lengthy explicit analysis.

- 2005, Hacon-M^{c}Kernan follows Shokurov's strategy & proves $\exists$ pl flips in all dim, assuming real MMP in dim one less.

  In fact, ① is the first of key steps in their proof.

Outline of proof of existence of flips:

$\exists$ flips

$\Big\Uparrow$ reduction (Shokurov)

$\exists$ pl flips

$\Big\Updownarrow$

f.g. of $R = R(X, K_X + S + B)$

$\Big\Updownarrow$ Lemma ($\star$)

Shokurov dim 2, 3 $\Rightarrow$ f.g. of restricted algebra R_S $\Leftarrow$ HM, all dim, assuming real MMP in dim one less

$\uparrow$

Main technical thm <u>4.3</u>

· What is a pl flip ?

· pl flipping contraction

$\|$

flipping contraction $\begin{array}{c}(X,\Delta)\\ f\downarrow \\ Z\end{array}$ with $\left\{\begin{array}{l} K_X+\Delta \text{ plt} \\ \quad \| \\ \underbrace{S}_{\text{coeff one irred}} + \underbrace{B}_{\text{fractional}} \end{array}\right.$

$\left\{\begin{array}{l} f \text{ small birat'l} \\ \quad \text{of } \rho(X/Z)=1 \\ -(K_X+\Delta) \ f\text{-ample} \end{array}\right.$

· pl flip = the flip of pl flipping contraction.

Thm (Shokurov) To prove the existence of flips it suffices to construct pl flips.

Recall (KM) $\exists$ flip of $D \underset{\substack{\uparrow \\ \text{pl flipping contraction}}}{=} K_X + S + B$

$\begin{array}{c} X \\ f\downarrow \\ Z \end{array}$

$\Updownarrow$

f.g. of $R(X,D) = \bigoplus_{n\geq 0} f_* \mathcal{O}_X(mD)$

§ 3. f. g. of R

<u>Goal</u> Prove f. g. of

$$R(X, K_X + S + B) = \bigoplus f_* \mathcal{O}_X(m(K_X + S + B))$$

coeff 1 ; fractional

ired

H^0

<u>Idea</u> Restrict to S & apply induction

Let $f: X \to Z = \operatorname{Spec} A$ proj morphism of normal varieties

<u>Def</u> Let $R = R(X, D)$, S: prime divisor.

The <u>restricted algebra</u>

$$R_S \overset{\text{defn}}{=} \operatorname{im}\Big(\underset{R}{\overset{\|}{\bigoplus_m f_* \mathcal{O}_X(mD)}} \xrightarrow{\text{restriction}} \bigoplus_m f_* \mathcal{O}_S(mD) \Big)$$

<u>Lemma</u> (☆) R f.g. $\Rightarrow$ R_S f.g.

— " — $\Leftarrow$ — " —

$$\begin{cases} S \sim dD, \ d \in \mathbb{N} \\ \text{where } D \geq 0 \ \& \ D \not\supset S \end{cases}$$

To prove (☆), need a basis

<u>Def + lemma</u> Let R = graded A-algebra

a <u>truncation</u> of R is

$$R_{(d)} = \bigoplus_{m \geq 0} R_{md}, \quad d \in \mathbb{N}$$

Then R is f.g. $\iff$ $R_{(d)}$ is f.g. some $d \in \mathbb{N}$.

[Ex. $R(X,D)$ f.g. $\iff$ $R(X,mD)$ f.g.]

Pf ($\Rightarrow$) $R_{(d)} =$ algebra of invariants $\left(\mathbb{Z}_d \curvearrowright R \right)$

$$\xi \in \mathbb{Z}_d,\ f \in R: \quad \xi \curvearrowright f = \xi^{\deg f} f$$

$\rightsquigarrow$ $R_{(d)}$ is f.g. by Noether's thm.

($\Leftarrow$) R is integral over $R_{(d)}$

($f \in R$ is a root of $x^d - f^d \in R_{(d)}[x]$)

$R_{(d)}$ f.g. $\Rightarrow$ R f.g. by Noether's thm on finiteness of integral closure.

Lemma (☆) R f.g. $\Rightarrow$ R_S f.g.

— " — $\Leftarrow$ — " — $\begin{cases} S \sim dD, \ d \in \mathbb{N} \\ \text{where } D \geq 0 \ \& \ D \not\geq S \end{cases}$

Pf. ($\Rightarrow$)

$$R \overset{\phi}{\underset{\text{restriction}}{\twoheadrightarrow}} R_S \quad \checkmark$$

($\Leftarrow$)

$$0 \to \boxed{\begin{matrix}\ker\phi \\ \text{f.g. ?}\end{matrix}} \to R \xrightarrow{\phi} \underset{\text{f.g.}}{R_S} \to 0$$

- Suppose $S \sim dD$.
- May assume $d=1$ by passing to a truncation: $R_{(d)} \longrightarrow (R_S)_{(d)}$.
- So $S = D + (g_1)$, $g_1 \in R_1$, by identifying $\left\{ \begin{matrix} \text{sections of} \\ R_m = f_* \mathcal{O}_X(mD) \end{matrix} \right\} \simeq \left\{ \begin{matrix} \text{rat'l functions } g \\ (g) + mD \geq 0 \end{matrix} \right\}$

Claim $\ker\phi = \langle g_1 \rangle$

Pf. Let $g \in R_m$ w/ $\phi(g) = 0$.

$\rightsquigarrow \operatorname{Supp}\big((g)+mD\big) \supset S$

$\rightsquigarrow (g)+mD = \underset{\substack{\| \\ D+(g_1)}}{S} + \underset{\substack{\vee\!\!/ \\ 0}}{S'}$

$\rightsquigarrow (g/g_1) + (m-1)D = S' \geq 0$

$\rightsquigarrow g/g_1 \in R_{m-1}$. ie. $g \in \langle g_1 \rangle$ ※

Now want f.g. of restricted alg. R_S.

In general $R_S \neq \bigoplus_m f_* \mathcal{O}_S(mD)$, but we'll show $\rightarrow$ ||

$$\bigoplus f_* \mathcal{O}_S(\underset{\substack{\uparrow \\ \text{additive}}}{B_m}) \quad \text{geometric}$$

Def. A geometric algebra is any $\mathcal{O}_Z$-algebra

$$R(X, B_\bullet) = \bigoplus_{m \geq 0} f_* \mathcal{O}_X(B_m)$$

where $B_\bullet$ is additive seq.

- a seq. of $\mathbb{R}$-divisors $B_\bullet$ is

additive if $B_i + B_j \leq B_{i+j}$

convex if $\frac{i}{i+j} B_i + \frac{j}{i+j} B_j \leq B_{i+j}$

note:

B_i additive

$\Updownarrow$

$\frac{B_i}{i}$ convex

Key pt f.g. of geometric algebra depends only on mobile part in each degree, even up to a birat'l map:

Lemma Let $Y \xrightarrow{g} X$ birat'l.

$R = R(X, B_\cdot)$, $R' = R(Y, B'_\cdot)$ geo. alg.

If $\mathrm{mob}(g^* B_m) = \mathrm{mob}(B'_m)$,

then $R \simeq R'$.

Pf.
$$R' = \underbrace{\bigoplus H^0(Y, \underset{\displaystyle \mathrm{mob}(g^* B_m)}{\mathrm{mob}(B'_m) \atop \|})}_{\displaystyle \underset{\displaystyle H^0(X, \mathrm{mob}(B_m))}{\cong}} \simeq R$$ ※

In fact, if mobile parts are free & "stabilize" $\Big\} \Rightarrow$ f.g. (next lemma)

Let $R = R(X, B_\cdot)$ geo. alg

Write $B_m = \underset{\text{mobile part}}{M_m} + \underset{\text{fixed part}}{F_m}$

Def. The seq. $M_\bullet$ is called mobile seq. $\to$ additive

" " $D_m = \frac{M_m}{m}$ " " characteristic seq. $\to$ convex

- R is semiample if $D = \lim D_m$ semiample.

Lemma R = semiample geo. alg on X.

If $D = D_k$, some $k > 0$,

then R is f.g.

Pf. Passing to a truncation, may assume $D = D_1$.

Then $mD = mD_1 = mM_1 \leq M_m = mD_m \leq mD$

So $M_m = mD$ semiample, all m.

Truncate further, may assume M_m free

Let $X \xrightarrow{h} W$ contraction s.t. $M_1 = h^* \underbrace{H}_{\text{v. ample}}$

Then $R = \bigoplus H^0(X, \underset{\substack{\| \\ mM_1 \\ \| \\ h^* mH}}{M_m}) = \bigoplus H^0(W, mH)$

f.g.

※

To apply this lemma, need

① improve how S sits inside X by changing models → semiample
(run real MMP)

② subtle saturation property → D_K stabilizes

§ 4. Main technical thm

<u>Thm 4.3</u> $\begin{array}{c}(X,\Delta)\\ {\scriptstyle f}\downarrow\\ Z\end{array}$ affine, $R = R(X, D)$, $D = k(K_X+\Delta)$ Cartier

Assume
(1) $K_X+\Delta$ plt
(2) $S = \lfloor \Delta \rfloor$ irred
(3) $\exists\, G \in |D|$, $\operatorname{Supp} G \not\ni S$
(4) $\Delta - S \sim_{\mathbb{Q}} \underbrace{A}_{\text{ample}} + \underbrace{B}_{\geq 0}$, $\operatorname{Supp} B \not\ni S$
(5) $-(K_X+\Delta)$ ample.

pl flipping contraction satisfies
1) f small birat'l, $\rho(X/Z)=1$, X: $\mathbb{Q}$-fac.
2) $K_X+\Delta$ plt, $S=\lfloor\Delta\rfloor$ irred
3) $-(K_X+\Delta)$ & $-S$ are ample

If real MMP holds in dim $n-1$, then R_S is f.g.

<u>Rk</u> The only interesting case = f birat'l
(otherwise, (5) $\Rightarrow$ $\kappa(X, K_X+\Delta) = -\infty$
$\rightarrow R_S = 0$)

<u>Lemma</u> 4.3 $\Rightarrow$ Main thm.

<u>Pf.</u> suffice to prove pl flipping contraction satisfies hypothesis of 4.3.

For (3) & (4),

Point : Every Cartier on X is mobile

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